Identifying system structure from controlled steady-state responses

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We suggest a control based method to uncover the structure (i.e., the dynamics of each system, the coupling direction, and coupling functions) of coupled systems. We show that driving a coupled system to steady states can reveal the underlying controlled coupling structure. An example of interacting quantum dots is presented to illustrate the structure identification method suggested.

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I. INTRODUCTION

Since the pioneering work of Pecora and Carroll [1], synchronization [2] of two coupled (chaotic) systems has been extensively investigated. In order to understand and predict the cooperative dynamic behavior of real coupled systems, one first has to identify the coupling structure (i.e., the dynamics of each system, the coupling direction, and coupling functions). In terms of the phase reduction theory [3], some authors [4–7] suggested to reveal the coupling direction which is applicable to understand functional relations between dynamical units in biological systems. However, these well-developed coupling direction detection methods [4–7] are not applicable to complex dynamical systems because the phase often cannot be well defined or is available, as well as synchronous systems because the information of coupling direction is hidden. Determination of coupling functions [8], on the other hand, has also attracted much attention. The coupling function estimation method [8] also depends on the phase reduction theory [3] and therefore, similarly as for coupling direction detection methods, it is not applicable to complex or synchronous dynamical systems. In this brief report, we suggest a control based approach to uncover the coupling structure, more precisely to estimate the dynamics of each system, coupling direction and coupling functions. This coupling structure identification method can be applied to complex or synchronous coupled systems.

We consider coupled systems, given by

$$\dot{\boldsymbol{x}}_1 = \mathbf{g}_1(\boldsymbol{x}_1) + \boldsymbol{h}_1(\boldsymbol{x}_1,\boldsymbol{x}_2) \triangleq \boldsymbol{f}_1(\boldsymbol{x}_1,\boldsymbol{x}_2), \tag{1}$$

$$\dot{\boldsymbol{x}}_2 = \boldsymbol{g}_2(\boldsymbol{x}_2) + \boldsymbol{h}_2(\boldsymbol{x}_1,\boldsymbol{x}_2) \triangleq \boldsymbol{f}_2(\boldsymbol{x}_1,\boldsymbol{x}_2), \tag{2}$$

where $\boldsymbol{x}_1 \in \mathbb{R}^N$ and $\boldsymbol{x}_2 \in \mathbb{R}^N$ (here $\mathbb{R}^j$ denotes j-dimensional real space) are state vectors of systems (1) and (2), respectively; $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ describe the node dynamics; $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ are coupling functions. We assume that when the coupling components are equal to each other, the coupling terms are equal to zero; more precisely, $\boldsymbol{h}_i(\boldsymbol{x},\boldsymbol{x})=0$ for all $\boldsymbol{x}$ and i. If $\boldsymbol{h}_1(\boldsymbol{x}_1,\boldsymbol{x}_2)=0$ for all $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, then no coupling exists from the second system to the first one. Otherwise, there exists a coupling from the second system to the first one. We can similarly determine the existence of coupling from the first system to the second one.

In order to introduce a more general formalism that can be applied to arbitrary systems, we shall summarize and reformulate Eqs. (1) and (2) now as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \tag{3}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector and $\boldsymbol{f}$ describes the system dynamics. Assume that $\boldsymbol{f}$ is Lipschitzian, that is, there exists a constant L such that $|(\boldsymbol{y}^T-\boldsymbol{x}^T)[\boldsymbol{f}(\boldsymbol{y})-\boldsymbol{f}(\boldsymbol{x})]| \leq L\|\mathbf{y}-\mathbf{x}\|^2$, where T is the transposed operator and $\|\cdots\|$ denotes the Euclidean norm. We first show that driving the system to steady states can be applied to estimate the structure, more precisely enables us to estimate the function $\boldsymbol{f}$; then we argue that when Eq. (3) describes the coupled system (1) and (2), this structure identification method can be used to estimate the dynamics of each system, coupling functions, and coupling direction.

II. THEORY

We add a control item

$$\boldsymbol{u} = -k(\boldsymbol{x}-\boldsymbol{z}) \tag{4}$$

to the right-hand side of Eq. (3), where k is the control gain and $\boldsymbol{z}$ is a constant vector to be specified. The following theorem (see the Appendix for its proof) provides the foundation for structure identification and gives the rules to design the control gain for driving the system (3) to steady state.

Theorem 1. When $\boldsymbol{u}$ has the form (4) with constant $\boldsymbol{z}$ freely chosen and with $k>2L+1$, the system (3) is driven to steady state s, satisfying

$$\boldsymbol{f}(\mathrm{s}) - k(\mathrm{s}-\boldsymbol{z}) = 0. \tag{5}$$

We now show that sufficiently many steady-state driving controls can be applied to identify function $\boldsymbol{f}$ from Eq. (5). When the mth driving control with constant $\boldsymbol{z}_m$ is performed, the resulting steady state s_m satisfies

$$\boldsymbol{f}(\mathbf{s}_m) = k(\mathbf{s}_m-\boldsymbol{z}_m), \tag{6}$$

which implies that we can estimate $\boldsymbol{f}(\mathbf{s}_m)$ by $k(\mathbf{s}_m-\boldsymbol{z}_m)$. After $n \gg 1$ driving controls are performed wherein each time $\boldsymbol{z}_m$ is gradually changed with a small enough rate in a desired range, we obtain n data pairs $\{\mathbf{s}_m, k(\mathbf{s}_m-\boldsymbol{z}_m)\}$ to represent the

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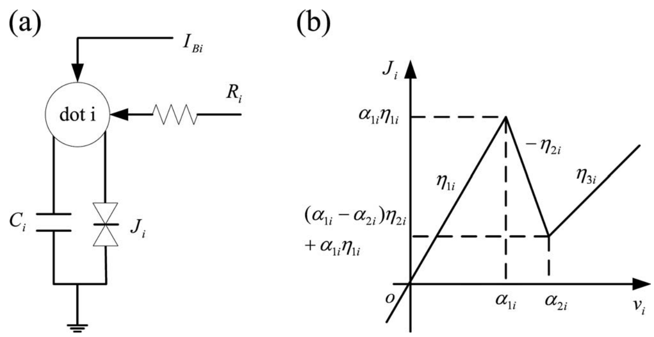

FIG. 1. (a) Quantum dot model. (b) Piecewise approximation of resonant tunneling diode.

input-output relation of the mapping $\boldsymbol{f}$ arbitrarily well by fitting methods.

We now come back to the structure identification issue of the coupled system (1) and (2). First, we can conclude from Eq. (5) that the steady-state-response equation of the coupled system (1) and (2) actually reads

$$\boldsymbol{f}_1(\mathbf{s}_1,\mathbf{s}_2)=\boldsymbol{g}_1(\mathbf{s}_1)+\boldsymbol{h}_1(\mathbf{s}_1,\mathbf{s}_2)=k(\mathbf{s}_1-z_1),$$

$$\boldsymbol{f}_2(\mathbf{s}_1,\mathbf{s}_2)=\boldsymbol{g}_2(\mathbf{s}_2)+\boldsymbol{h}_2(\mathbf{s}_1,\mathbf{s}_2)=k(\mathbf{s}_2-\mathbf{z}_2). \tag{7}$$

As analyzed above, functions $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ can be estimated with high accuracy when sufficiently many driving controls are performed in which each time z_1 and z_2 are gradually changed with a small enough rate in a desired range.

Substituting $\mathbf{s}_1=\mathbf{s}_2=\mathbf{d}$ into Eq. (7) and noting $\boldsymbol{h}_i(\boldsymbol{x},\boldsymbol{x})=0$ for all $\boldsymbol{x}$ and i, we obtain

$$\boldsymbol{g}_1(\mathbf{d})=\boldsymbol{f}_1(\mathbf{d},\mathbf{d}),g_2(\mathbf{d})=\boldsymbol{f}_2(\mathbf{d},\mathbf{d}).$$

This implies that the intersection curve between surfaces $\mathbf{s}_1=\mathbf{s}_2$ and $\boldsymbol{f}_i(\mathbf{s}_1,\mathbf{s}_2)$ can be applied to approximate the function $\boldsymbol{g}_i$. Furthermore, the function $\boldsymbol{h}_i$ can finally be approximated from the estimated $\boldsymbol{f}_i(\mathbf{s}_1,\mathbf{s}_2)-\boldsymbol{g}_i(\mathbf{s}_i)$. If the estimated $\boldsymbol{h}_1(\mathbf{s}_1,\mathbf{s}_2)$ approximates to zero for any $\mathbf{s}_1$ and $\mathbf{s}_2$, then no coupling exists from the second system to the first one; otherwise, there exists a coupling from the second system to the first one. Similarly, if the estimated $\boldsymbol{h}_2(\mathbf{s}_1,\mathbf{s}_2)$ approximates to zero for any $\mathbf{s}_1$ and $\mathbf{s}_2$, then no coupling exists from the first system to the second one; otherwise, there exists a coupling from the first system to the second one.

III. EXAMPLES

To illustrate the structure identification method suggested, we treat two resistively coupled quantum dots (see Ref. [9], and references therein) whose current equations [see Fig. 1(a)] are described as

$$C_1\dot{v}_1=I_{B1}-J_1(v_1)+R_1(v_2-v_1), \tag{8}$$

$$C_2\dot{v}_2=I_{B2}-J_2(v_2)+R_2(v_1-v_2), \tag{9}$$

where v_i is the voltage of the dot i, I_{Bi} is the external bias current of the dot i, $J_i(v_i)$ is the current of the resonant tunneling diode (RTD), C_i is the capacitance between the dot island and the substrate, R_i is the coupling resistance. As illustrated in Ref. [10], coupled quantum dots can generate wave propagation that is applicable to some data processing (such as analog-to-digital conversion). The wave propagation behavior of coupled quantum dots is mainly determined by the RTD model of each quantum dot, the external bias currents, and the coupling resistances. To design and predict dynamic behavior of coupled quantum dots, we first have to identify these structural parameters. This will be achieved below using the structure identification method suggested. In the following simulations, the RTD of each quantum dot is modeled by a piecewise linear function shown in Fig. 1(b). Furthermore, $C_1=C_2=1$, $I_{B1}=I_{B2}=0$, $R_1=R_2=0.1$, $\alpha_{11}=0.2$, $\alpha_{12}=0.8$, $\eta_{11}=0.25$, $\eta_{12}=0.25$, $\eta_{12}=0.5$, $\alpha_{21}=0.25$, $\alpha_{22}=0.76$, $\eta_{21}=0.2$, $\eta_{22}=0.2$, and $\eta_{i2}=0.6$. However, we assume that we have no prior knowledge about RTD models and we assume that coupling terms have the general form $h_i(v_1,v_2)$ for all $i=1,2$ and $h_i(x,x)=0$ for all x and i.

We add the control terms

$$u_1=-k(v_1-\theta_1), \quad u_2=-k(v_2-\theta_2) \tag{10}$$

to Eqs. (8) and (9), respectively. By Theorem 1, for any θ_1 and θ_2, the coupled system (8) and (9) is driven to a steady state (s_1,s_2) (depending on θ_i and k), satisfying

$$f_i(s_1,s_2)\triangleq-J_i(s_i)+h_i(s_1,s_2)=k(s_i-\theta_i) \tag{11}$$

for all $i=1,2$.

Letting (θ_1,θ_2) traverse the area $[-2,2]\times[-2,2]$ with rate 0.1 per step along each axis, we measure the resulting steady state (s_1,s_2) and scan the value of f_i at (s_1,s_2) using Eq. (11), as illustrated in Fig. 2 for which $k=10$. It is easy to see from Fig. 2(c) that function f_1 is estimated with high accuracy. We have validated that function f_2 can also be estimated with high accuracy (not shown for compactness).

Noting Eq. (11) and $h_i(x,x)=0$ for all x, we can identify RTD model J_1 by achieving the intersection curve between surfaces $s_1=s_2$ and the estimated $-f_1(s_1,s_2)$, as illustrated in Fig. 3. Similarly, we obtain the intersection curve between surfaces $s_1=s_2$ and the estimated $-f_2(s_1,s_2)$ to approximate the RTD model J_2 (not shown for compactness).

We can conclude from Eq. (11) that function h_i can be approximated from the estimated $f_i(s_1,s_2)+J_i(s_i)$, as shown in Fig. 4. It is easy to see from Fig. 4(b) that h_1 can be estimated with high accuracy. This implies that there exists a coupling from the second quantum dot to the first one. Similarly, we can identify h_2 and determine the existence of coupling from the first quantum dot to the second one (not shown for compactness).

Finally, we assume that two uniformly distributed noises with amplitude ranging from -0.01 to 0.01 are added to the right-hand side of Eqs. (8) and (9), respectively, and we consider the influence of noise on structure identification. As a typical result Fig. 5 summarizes our results. It is easy to see from Fig. 5(b) that the structure identification method suggested is robust and we can still estimate function f_1 with high accuracy even in the presence of noise.

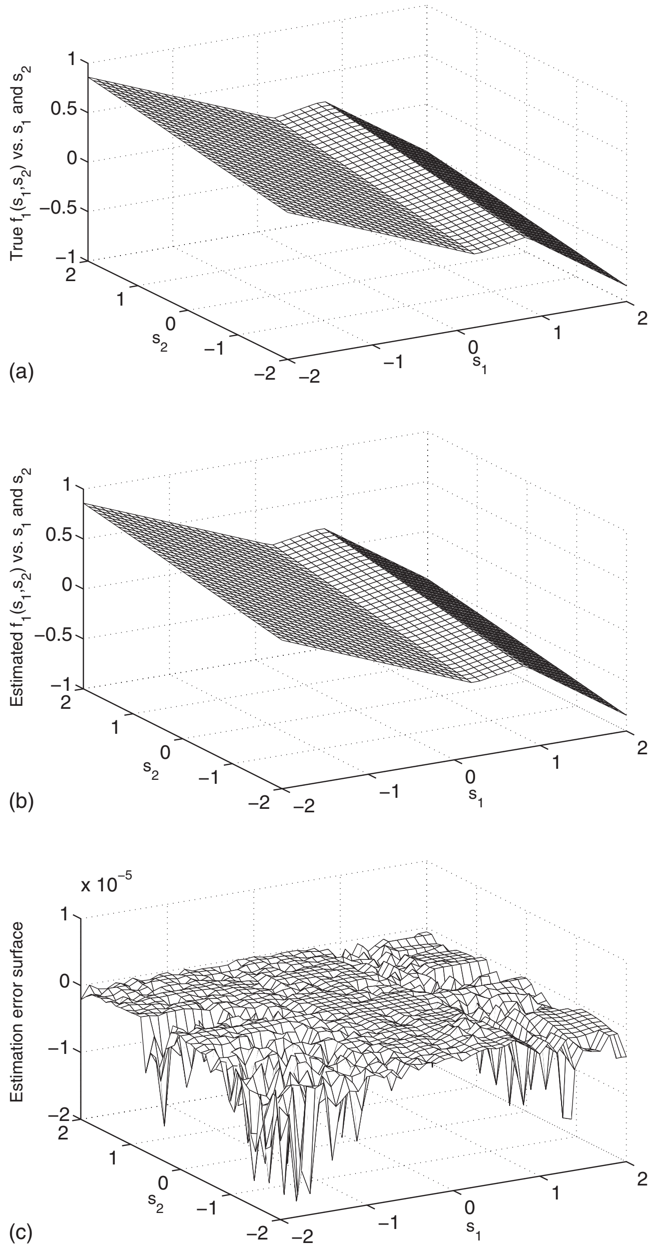

FIG. 2. Identification of function f_1. (a) True f_1. (b) Estimated f_1. (c) Estimation error.

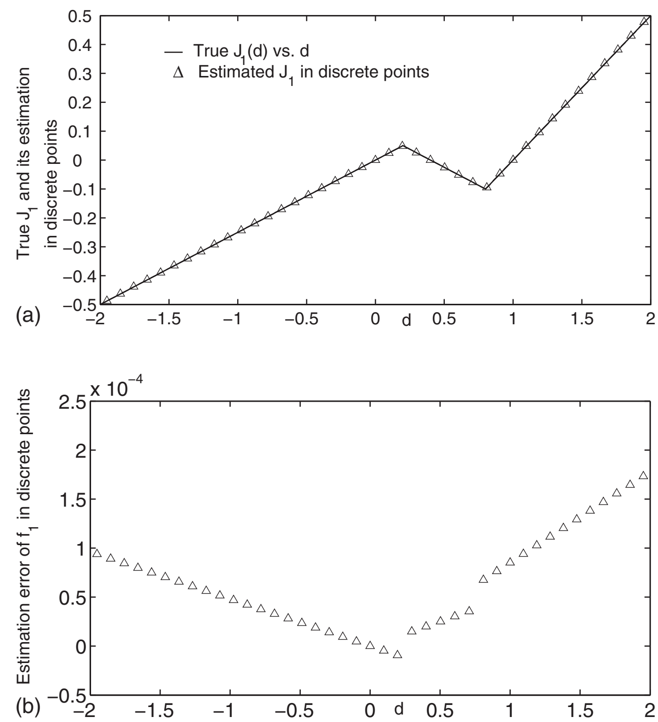

FIG. 3. RTD model identification. (a) True RTD model J_1 (solid line) and its estimation (Δ) in discrete points. (b) Estimation error of J_1 in discrete points shown in (a).

IV. CONCLUSIONS

We showed that driving a system to steady states can reveal the underlying controlled structure. This structure identification method can be applied to uncover the coupling structure (i.e., the dynamics of individual nodes, the coupling direction, and coupling functions) of coupled systems.

Recently the study of dynamical networks [11–13] has attracted increasing interest within the nonlinear science community. The current research focused on understanding the emerging cooperative phenomena as well as uncovering the relation between structure and functions of various real networks such as neurons, power stations, interacting genes, or coupled nonlinear oscillators. In order to understand the emerging cooperative dynamic behavior and then the emergent functions of a real network, one first has to identify the coupling structure (i.e., the dynamics of individual nodes, the coupling functions, and the connection topology) [13]. We are now investigating how to extend this structure identification method to dynamical networks who support the required steady-state driving control (at least within a short time [14]), such as quantum-dot networks [9] and excitable media. We are also studying if it is possible to reveal system structure by driving systems to given periodical trajectories.

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APPENDIX: PROOF OF THEOREM 1

We first prove the existence of $\mathbf{s}$, satisfying Eq. (5), through the following lemma.

Lemma 1(Th.1[15]*)*. Let $\boldsymbol{g}(\cdots):I\mathbb{R}^n \to I\mathbb{R}^n$ be continuous. Let $M \geq 0$, $a>0$, b and c be real numbers such that

$$\boldsymbol{x}^T[\boldsymbol{g}(\boldsymbol{x})-\boldsymbol{y}] \geq a\|\mathbf{x}\|^2 + b\|\mathbf{x}\| + c, \quad \text{if} \|\mathbf{x}\| \geq M. \tag{12}$$

Then $\boldsymbol{g}(\boldsymbol{x})=\boldsymbol{y}$ has a solution in a ball $B(0,r):=\{\boldsymbol{x} \| \|\mathbf{x}\| \leq r\}$, where we have the following.

(A) $r=M$, if (i) $(b/2a)^2 - c/a \leq 0$ or (ii) $(b/2a)^2 - c/a > 0$ and $M \leq -b/2a - \sqrt{(b/2a)^2 - c/a}$.

(B) $r=\max[M, -b/2a+\sqrt{(b/2a)^2 - c/a}]$, if $(b/2a)^2 - c/a > 0$ and $-b/2a - \sqrt{(b/2a)^2 - c/a} < M$. ■

It is easy to see

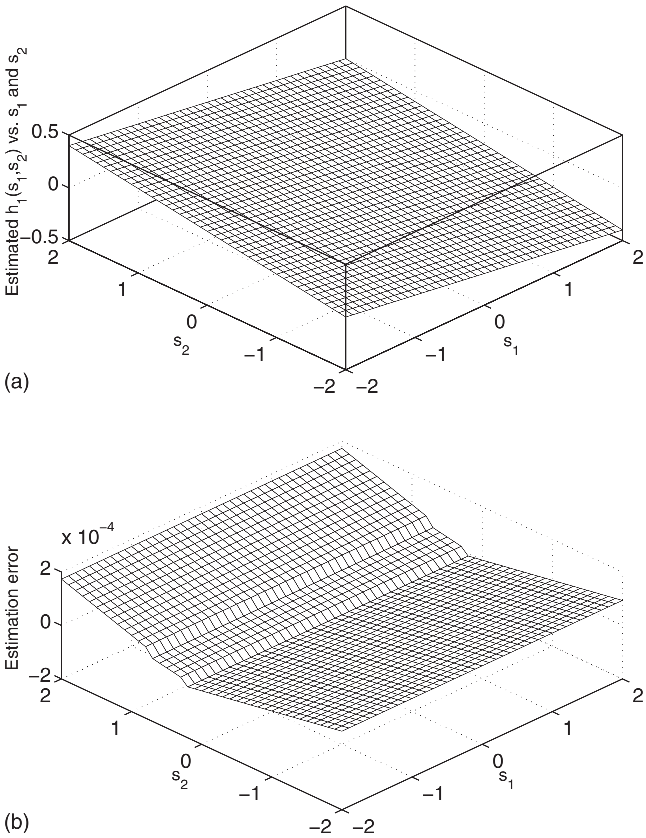

FIG. 4. Identification of coupling function $h_1(s_1,s_2)=0.1(s_2-s_1)$. (a) Estimated $h_1(s_1,s_2)$ versus s_1 and s_2. (b) Estimation error surface.

$$
\begin{aligned}
\mathbf{s}^T[k(\mathbf{s}-z)-\boldsymbol{f}(\mathbf{s})] &= \mathbf{s}^T[k\mathbf{s}-\boldsymbol{f}(\mathbf{s})-kz] \\
&= k\mathbf{s}^T\mathbf{s}-\mathbf{s}^T[\boldsymbol{f}(\mathbf{s})-\boldsymbol{f}(0)]-\mathbf{s}^Tkz-\mathbf{s}^T\boldsymbol{f}(0) \\
&\geq (k-L)\|\mathbf{s}\|^2-(1/2)[(k+1)\|\mathbf{s}\|^2+k\|\mathbf{z}\|^2+\|\boldsymbol{f}(0)\|^2] \\
&= (k/2-L-1/2)\|\mathbf{s}\|^2-(1/2)[k\|\mathbf{z}\|^2+\|\boldsymbol{f}(0)\|^2].
\end{aligned}
$$

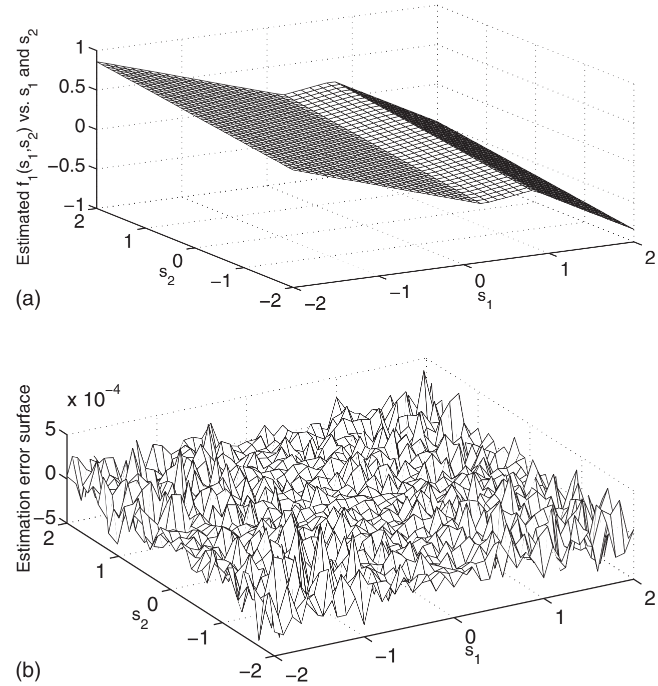

FIG. 5. Identification of function f_1 in the presence of noise. (a) Estimated f_1 [true f_1 has been plotted in Fig. 2(a)]. (b) Estimation error surface.

It follows from lemma 1 that there exists a solution in the ball $B[0,\sqrt{(k\|\mathbf{z}\|^2+\|\boldsymbol{f}(0)\|^2)/(k-2L-1)}]$ when $k>2L+1$.

Next, we analyze the stability of the steady state. Let $\mathbf{e}=\boldsymbol{x}-\mathbf{s}$. Then we get

$$\dot{\mathbf{e}}=\boldsymbol{f}(\mathbf{e}+\mathbf{s})-\boldsymbol{f}(\mathbf{s})-k\mathbf{e}. \tag{13}$$

Choosing a Lyapunov function $V=\mathbf{e}^T\mathbf{e}/2$, we have

$$\dot{V}=\mathbf{e}^T[\boldsymbol{f}(\mathbf{e}+\mathbf{s})-\boldsymbol{f}(\mathbf{s})]-k\mathbf{e}^T\mathbf{e}\leq-(k-L)\mathbf{e}^T\mathbf{e},$$

which indicates that when $k>L$, $\mathbf{e}\to 0$ and thereby the theorem is proved.

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